

Optimal quadrature formulas with derivatives in Sobolev space

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Abstract

In the present paper the problem of construction of optimal quadrature formulas in the sense of Sard in the space $L_2^{(m)}(0, 1)$ is considered. Here the quadrature sum consists of values of the integrand at nodes and values of the first and the third derivatives of the integrand at the end points of the integration interval. The coefficients of optimal quadrature formulas are found and the norm of the optimal error functional is calculated for arbitrary natural number N and for any $m \geq 4$ using S.L. Sobolev method which is based on discrete analogue of the differential operator d^{2m}/dx^{2m} . In particular, for $m = 4, 5$ optimality of the classical Euler-Maclaurin quadrature formula is obtained. Starting from $m = 6$ new optimal quadrature formulas are obtained.

Key words: optimal quadrature formulas, the error functional, the extremal function, S.L. Sobolev space, optimal coefficients.

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1 Introduction. Statement of the problem

We consider the following general quadrature formula

$$\int_0^1 p(x) \varphi(x) dx \cong \sum_{\beta=0}^N \sum_{j=0}^{\alpha} C_{\beta j} \varphi^{(j)}(x_{\beta}) \quad (1.1)$$

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with the error functional

$$\ell(x) = p(x)\varepsilon_{[0,1]}(x) - \sum_{\beta=0}^N \sum_{j=0}^{\alpha} (-1)^j C_{\beta j} \delta^{(j)}(x - x_{\beta}) \quad (1.2)$$

in a Banach space B . Here $C_{\beta j}$ are the coefficients and x_{β} are the nodes of the formula (1.1), $N = 1, 2, \dots$, $\alpha = 0, 1, \dots$, $p(x)$ is a weight function, $\varepsilon_{[0,1]}(x)$ is the characteristic function of the interval $[0,1]$, $\delta(x)$ is the Dirac delta-function, φ is an element of the space B .

The difference

$$(\ell, \varphi) = \int_{-\infty}^{\infty} \ell(x)\varphi(x)dx = \int_0^1 p(x)\varphi(x)dx - \sum_{\beta=0}^N \sum_{j=0}^{\alpha} C_{\beta j} \varphi^{(j)}(x_{\beta}) \quad (1.3)$$

is called *the error* of the quadrature formula (1.1).

By the Cauchy-Schwarz inequality

$$|(\ell, \varphi)| \leq \|\varphi|B\| \cdot \|\ell|B^*\|$$

the error (1.3) of the formula (1.1) is estimated with the help of the norm of the error functional (1.2) in the conjugate space B^* , i.e. by

$$\|\ell|B^*\| = \sup_{\|\varphi|B\|=1} |(\ell, \varphi)|.$$

Thus estimation of the error (1.3) of the quadrature formula (1.1) on functions of the space B is reduced to finding the norm of the error functional ℓ in the conjugate space B^* .

Obviously the norm of the error functional ℓ depends on the coefficients and the nodes of the quadrature formula (1.1). The problem of finding the minimum of the norm of the error functional ℓ by coefficients and by nodes is called *S.M. Nikol'skii problem*, and obtained formula is called *optimal quadrature formula in the sense of Nikol'skii*. This problem was first considered by S.M. Nikol'skii [17], and continued by many authors, see e.g. [2,3,4,5,18,38] and references therein. Minimization of the norm of the error functional ℓ by coefficients when the nodes are fixed is called *Sard's problem*. And obtained formula is called *optimal quadrature formula in the sense of Sard*. First this problem was investigated by A.Sard [19].

The results of this paper are related to Sard's problem. So here we discuss some of the previous results about optimal quadrature formulas in the sense of Sard which are closely connected to our results.

There are several methods of construction of optimal quadrature formulas in the sense of Sard such as spline method, φ -function method (see e.g. [2,21])

and Sobolev's method which is based on construction of discrete analogue of a linear differential operator (see e.g. [34,35]). In the different spaces, based on these methods, the Sard's problem was investigated by many authors, see, for example, [1,2,4,6,7,9,11,12,13,14,15,16,20,21,22,23,25,28,29,30,31,32,33,34,35,36,37] and references therein.

In the paper [21], using spline method, optimality of the classical Euler-Maclaurin formula was proved and the error of this quadrature formula is calculated in $L_2^{(m)}(0, n)$, where $L_2^{(m)}(0, n)$ is the space of functions which are square integrable with m -th generalized derivative.

Let $W_{L_p}^m$ ($m = 1, 2, \dots, 1 \leq p \leq \infty$) be a class of functions f , having on the $[0,1]$ $(m-1)$ - absolute continues derivative and $\|f^{(m)}\|_p \leq 1$, where $\|\cdot\|_p = \|\cdot\|_{L_p(0,1)}$. In [38] it is proved, that among quadrature formulas (1.1) when $p(x) = 1$ the Euler-Maclaurin quadrature formula is optimal in the space $W_{L_p}^m$. And in [27] optimality of the lattice cubature formulas of Euler-Maclaurin type is proved in the space $L_2^{(m)}$.

Using φ -function method optimality of the Euler-Maclaurin quadrature formula is proved and the error of this formula is calculated by T. Catinaş and Gh. Coman [4] in the space $L_2^{(2)}(0, 1)$. Also using this method in [14] a procedure of construction of quadrature formulas of the form (1.1), which are exact for solutions of linear differential equations and are optimal in the sense of Sard is discussed.

It should be noted, that in applications the formula (1.1) is interesting for small values of α . Optimal quadrature formulas in the sense of Sard for the case $\alpha = 0$ has already been discussed by many authors, mainly in the space $L_2^{(m)}$ (see [1,2,4,6,7,9,11,13,14,15,16,20,21,22,23,25,28,29,33,34,35,36,37] and references therein).

The main aim of this paper is to construct optimal quadrature formulas of the form (1.1) in the sense of Sard for the case $\alpha = 3$ when $p(x) = 1$ in the space $L_2^{(m)}(0, 1)$ equipped with the norm

$$\|\varphi(x)\|_{L_2^{(m)}(0,1)} = \left\{ \int_0^1 (\varphi^{(m)}(x))^2 dx \right\}^{1/2}$$

and $\int_0^1 (\varphi^{(m)}(x))^2 dx < \infty$.

We use the Sobolev method [34,35] which is based on the discrete analogue of the differential operator d^{2m}/dx^{2m} . We consider the following quadrature

formula

$$\int_0^1 \varphi(x) dx \cong \sum_{\beta=0}^N C[\beta] \varphi[\beta] + A \left(\varphi'(0) - \varphi'(1) \right) + B \left(\varphi'''(0) - \varphi'''(1) \right) \quad (1.4)$$

with the error functional

$$\ell(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^N C[\beta] \delta(x - h\beta) + A \left(\delta'(x) - \delta'(x-1) \right) + B \left(\delta'''(x) - \delta'''(x-1) \right) \quad (1.5)$$

in the space $L_2^{(m)}(0,1)$ for $m \geq 4$. Here $C[\beta]$, $\beta = \overline{0, N}$, A and B are the coefficients of the formula (1.4), $h = \frac{1}{N}$, N is a natural number.

For the error functional (1.5) to be defined on the space $L_2^{(m)}(0,1)$ it is necessary to impose the following conditions (see [33])

$$(\ell(x), x^\alpha) = 0, \quad \alpha = 0, 1, 2, \dots, m-1. \quad (1.6)$$

Hence it is clear that for existence of the quadrature formulas of the form (1.4) the condition $N \geq m-3$ has to be met.

Note that here in after ℓ means the functional (1.5).

As was noted above by the Cauchy-Schwarz inequality, the error of the formula (1.4) is estimated by the norm $\|\ell|L_2^{(m)*}(0,1)\|$ of the error functional (1.5). Furthermore the norm of the error functional (1.5) depends on the coefficients $C[\beta]$, A and B . We minimize the norm of the error functional (1.5) by the coefficients $C[\beta]$, A and B , i.e., we find

$$\left\| \overset{\circ}{\ell} | L_2^{(m)*} \right\| = \inf_{C[\beta], A, B} \left\| \ell | L_2^{(m)*} \right\|. \quad (1.7)$$

The coefficients $C[\beta]$, A and B which satisfy the equality (1.7) is called *the optimal coefficients* and denoted by $\overset{\circ}{C}[\beta]$, $\overset{\circ}{A}$ and $\overset{\circ}{B}$ and the corresponding quadrature formula is called *the optimal quadrature formula in the sense of Sard*. In the sequel, for the purposes of convenience the optimal coefficients $\overset{\circ}{C}[\beta]$, $\overset{\circ}{A}$ and $\overset{\circ}{B}$ will be denoted as $C[\beta]$, A and B .

Thus to construct optimal quadrature formulas in the form (1.4) in the sense of Sard we have to consequently solve the following problems.

Problem 1. Find the norm of the error functional (1.5) of the quadrature formula of the form (1.4) in the space $L_2^{(m)*}(0,1)$.

Problem 2. Find coefficients $C[\beta]$, A and B which satisfy the equality (1.7).

The paper is organized as follows. In section 2 we give some definitions and known formulas. In section 3 we determine the extremal function which corresponds to the error functional ℓ and give a representation of the norm of the error functional (1.5). Section 4 is devoted to a minimization of $\|\ell\|^2$ with respect to the coefficients $C[\beta]$, A and B . We obtain a system of linear equations for the coefficients of the optimal quadrature formula of the form (1.4) in the sense of Sard in the space $L_2^{(m)}(0, 1)$. Explicit formulas for coefficients of the optimal quadrature formula of the form (1.4) are found in subsection 5.1. Moreover we calculate the norm of the error functional (1.5) of the optimal quadrature formula of the form (1.4) in subsection 5.2.

2 Definitions and known formulas

In this section we give some definitions and formulas that we need to prove the main results.

Here the main concept used is that of functions of discrete arguments and operations on them (see. [33,35]). For the purposes of completeness we give some definitions about functions of discrete argument.

Assume that φ and ψ are real-valued functions of real variable and are defined in real line \mathbb{R} .

Definition 2.1. Function $\varphi(h\beta)$ is called *function of discrete argument*, if it is given on some set of integer values of β .

Definition 2.2. The *inner product* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ is called the number

$$[\varphi, \psi] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta),$$

if the series on the right hand side of the last equality converges absolutely.

Definition 2.3. The *convolution* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ is called the inner product

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma).$$

The *Euler-Frobenius polynomials* $E_k(x)$, $k = 1, 2, \dots$ is defined by the following

formula [35]

$$E_k(x) = \frac{(1-x)^{k+2}}{x} \left(x \frac{d}{dx} \right)^k \frac{x}{(1-x)^2}, \quad (2.1)$$

$$E_0(x) = 1.$$

For the Euler-Frobenius polynomials $E_k(x)$ the following identity holds

$$E_k(x) = x^k E_k\left(\frac{1}{x}\right), \quad (2.2)$$

and also the following theorem is true

Theorem 2.1 (Lemma 3 of [24]). Polynomial $Q_k(x)$ which is defined by the formula

$$Q_k(x) = (x-1)^{k+1} \sum_{i=0}^{k+1} \frac{\Delta^i 0^{k+1}}{(x-1)^i} \quad (2.3)$$

is the Euler-Frobenius polynomial (2.1) of degree k , i.e. $Q_k(x) = E_k(x)$, where $\Delta^i 0^k = \sum_{l=1}^i (-1)^{i-l} C_i^l l^k$.

The following formula is valid [10]:

$$\sum_{\gamma=0}^{n-1} q^\gamma \gamma^k = \frac{1}{1-q} \sum_{i=0}^k \left(\frac{q}{1-q} \right)^i \Delta^i 0^k - \frac{q^n}{1-q} \sum_{i=0}^k \left(\frac{q}{1-q} \right)^i \Delta^i \gamma^k|_{\gamma=n}, \quad (2.4)$$

where $\Delta^i \gamma^k$ is the finite difference of order i of γ^k , q is ratio of a geometric progression.

At last we give the following well known formulas from [8]

$$\sum_{\gamma=0}^{\beta-1} \gamma^k = \sum_{j=1}^{k+1} \frac{k! B_{k+1-j}}{j! (k+1-j)!} \beta^j, \quad (2.5)$$

where B_{k+1-j} are Bernoulli numbers,

$$\Delta^\alpha x^\nu = \sum_{p=0}^{\nu} C_\nu^p \Delta^\alpha 0^p x^{\nu-p}. \quad (2.6)$$

3 The extremal function and the representation of the error functional norm

To solve Problem 1, i.e., for finding the norm of the error functional (1.5) in the space $L_2^{(m)}(0,1)$ a concept of the extremal function is used [33]. The function ψ_ℓ is said to be *the extremal function* of the error functional (1.5) if

the following equality holds

$$(\ell, \psi_\ell) = \|\ell|L_2^{(m)*}\| \|\psi_\ell|L_2^{(m)}\|. \quad (3.1)$$

In the space $L_2^{(m)}$ the extremal function ψ_ℓ of a functional ℓ was found by S.L. Sobolev [33,35]. This extremal function has the form

$$\psi_\ell(x) = (-1)^m \ell(x) * G(x) + P_{m-1}(x), \quad (3.2)$$

where

$$G(x) = \frac{|x|^{2m-1}}{2 \cdot (2m-1)!} \quad (3.3)$$

is a solution of the equation

$$\frac{d^{2m}}{dx^{2m}} G(x) = \delta(x), \quad (3.4)$$

$P_{m-1}(x)$ is a polynomial of degree $m-1$, the symbol $*$ is operation of convolution, i.e.

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy = \int_{-\infty}^{\infty} f(y)g(x-y)dy.$$

It is well known that for any functional ℓ in $L_2^{(m)*}$ the equality

$$\begin{aligned} \|\ell|L_2^{(m)*}(0,1)\|^2 &= (\ell, \psi_\ell) = (\ell(x), (-1)^m \ell(x) * G(x)) = \\ &= \int_{-\infty}^{\infty} \ell(x) \left((-1)^m \int_{-\infty}^{\infty} \ell(y) G(x-y) dy \right) dx \end{aligned}$$

holds [33].

Applying this equality to the error functional (1.5) we obtain the following

$$\begin{aligned} \|\ell\|^2 &= (\ell, \psi_\ell) = \\ &= (-1)^m \left[\sum_{\beta=0}^N \sum_{\gamma=0}^N C[\beta]C[\gamma] \frac{|h\beta - h\gamma|^{2m-1}}{2(2m-1)!} - 2 \sum_{\beta=0}^N C[\beta] \int_0^1 \frac{|x - h\beta|^{2m-1}}{2(2m-1)!} dx - \right. \\ &\quad - 2A \sum_{\beta=0}^N C[\beta] P_{2m-2}[\beta] - 2B \sum_{\beta=0}^N C[\beta] P_{2m-4}[\beta] + \frac{2A}{(2m-1)!} + \\ &\quad \left. + \frac{2B + A^2}{(2m-3)!} + \frac{2AB}{(2m-5)!} + \frac{B^2}{(2m-7)!} - \frac{1}{(2m+1)!} \right]. \quad (3.5) \end{aligned}$$

where $P_k(h\beta) = \frac{(h\beta)^k + (1-h\beta)^k}{2k!}$.

Thus Problem 1 is solved for quadrature formulas of the form (1.4) in the space $L_2^{(m)}(0, 1)$.

4 The system for optimal coefficients of the quadrature formula (1.4)

Now we investigate Problem 2. For finding the minimum of $\|\ell\|^2$ under the conditions (1.6) the Lagrange method is used. For this we consider the following function

$$\Psi = \|\ell\|^2 - 2 \cdot (-1)^m \sum_{\alpha=0}^{m-1} \lambda_\alpha (\ell(x), x^\alpha),$$

where λ_α are unknown multipliers. The function Ψ is the multidimensional function with respect to the coefficients $C[\beta]$, A , B and λ_α . Equating to zero partial derivatives of the Ψ by coefficients $C[\beta]$, A and B , together with the conditions (1.6) we get the following system of linear equations

$$\sum_{\gamma=0}^N C[\gamma] \frac{|h\beta - h\gamma|^{2m-1}}{2(2m-1)!} - AP_{2m-2}(h\beta) - BP_{2m-4}(h\beta) + \sum_{\alpha=0}^{m-1} \lambda_\alpha (h\beta)^\alpha = f_m(h\beta), \quad \beta = \overline{0, N}, \quad (4.1)$$

$$\sum_{\beta=0}^N C[\beta] P_{2m-2}(h\beta) - \frac{A}{(2m-3)!} - \frac{B}{(2m-5)!} + \sum_{\alpha=2}^{m-1} \alpha \cdot \lambda_\alpha = \frac{1}{(2m-1)!}, \quad (4.2)$$

$$\begin{aligned} \sum_{\beta=0}^N C[\beta] P_{2m-4}(h\beta) - \frac{A}{(2m-5)!} - \frac{B}{(2m-7)!} + \\ + \sum_{\alpha=4}^{m-1} \alpha(\alpha-1)(\alpha-2) \cdot \lambda_\alpha = \frac{1}{(2m-3)!}, \end{aligned} \quad (4.3)$$

$$\sum_{\beta=0}^N C[\beta] (h\beta)^i = \frac{1}{i+1}, \quad i = \overline{0, 1} \quad (4.4)$$

$$\sum_{\beta=0}^N C[\beta] (h\beta)^j - jA = \frac{1}{j+1}, \quad j = \overline{2, 3} \quad (4.5)$$

$$\sum_{\beta=0}^N C[\beta] (h\beta)^\alpha - \alpha A - \alpha(\alpha-1)(\alpha-2)B = \frac{1}{\alpha+1}, \quad \alpha = \overline{4, m-1}. \quad (4.6)$$

where $P_k(h\beta) = \frac{(h\beta)^k + (1-h\beta)^k}{2k!}$.

The system (4.1)-(4.6) is called *the discrete system of Wiener-Hopf type* for the optimal coefficients [33,35]. In the system (4.1)-(4.6) the coefficients $C[\beta]$, $\beta = \overline{0, N}$, A and B , and also λ_α , $\alpha = \overline{0, m-1}$ are unknowns. The system

(4.1)-(4.6) has unique solution and this solution gives the minimum to the $\|\ell\|^2$. Here we omitted the proof of the existence and uniqueness of the solution of the system (4.1) - (4.6). The proof of the existence and uniqueness of the solution of this system is as the proof of the existence and uniqueness of the solution of discrete Wiener-Hopf type system of the optimal coefficients in the space $L_2^{(m)}(0, 1)$ for quadrature formulas of the form (1.1) for the case $\alpha = 0$ (see [33,35]). It should be noted, that in [14] the uniqueness of the optimal quadrature formulas in the Sard's sense of the form (1.1) is discussed.

5 The coefficients and the norm of the error functional of the optimal quadrature formulas

In the present section we study the solution of the system (4.1)-(4.6). To solve this system we use the approach which was suggested by S.L. Sobolev in [34].

5.1 The coefficients of the optimal quadrature formulas of the form (1.4)

Suppose that $C[\beta] = 0$ for $\beta < 0$ and $\beta > N$. Using Definition 2.3 we rewrite the equation (4.1) in the convolution form:

$$C[\beta] * \frac{|h\beta|^{2m-1}}{2(2m-1)!} - AP_{2m-2}(h\beta) - BP_{2m-4}(h\beta) + \sum_{\alpha=0}^{m-1} \lambda_{\alpha}(h\beta)^{\alpha} = f_m(h\beta), \quad \beta = \overline{0, N}, \quad (5.1)$$

where

$$f_m(h\beta) = \int_0^1 \frac{|x - h\beta|^{2m-1}}{2(2m-1)!} dx = \frac{(h\beta)^{2m}}{(2m)!} + \sum_{j=0}^{2m-1} \frac{(-h\beta)^{2m-1-j}}{2(2m-1-j)! \cdot (j+1)!}. \quad (5.2)$$

We consider the following problem.

Problem A. Find the discrete function $C[\beta]$ and unknown coefficients A , B , λ_{α} , which satisfy the system (4.1)-(4.6).

Further, instead of $C[\beta]$ we introduce the functions

$$v(h\beta) = C[\beta] * \frac{|h\beta|^{2m-1}}{2(2m-1)!}, \quad (5.3)$$

$$u(h\beta) = v(h\beta) - AP_{2m-2}(h\beta) - BP_{2m-4}(h\beta) + \sum_{\alpha=0}^{m-1} \lambda_{\alpha}(h\beta)^{\alpha}. \quad (5.4)$$

In this statement it is necessary to express $C[\beta]$ by the function $u(h\beta)$. For this we need such operator $D_m(h\beta)$, which satisfies the equation

$$hD_m(h\beta) * G(h\beta) = \delta(h\beta), \quad (5.5)$$

where $G(h\beta) = \frac{|h\beta|^{2m-1}}{2(2m-1)!}$ is the discrete argument function corresponding to $G(x)$ defined by (3.3), $\delta(h\beta)$ is equal to 0 when $\beta \neq 0$ and is equal to 1 when $\beta = 0$, i.e. $\delta(h\beta)$ is the discrete delta-function. The equation (5.5) is the discrete analogue of equation (3.4). So the discrete function $D_m(h\beta)$ is called *the discrete analogue* of the differential operator d^{2m}/dx^{2m} [33].

It should be noted that the operator $D_m(h\beta)$ was firstly introduced and investigated by S.L. Sobolev [33].

In [26] the discrete analogue $D_m(h\beta)$ of the differential operator d^{2m}/dx^{2m} , which satisfies equation (5.5), is constructed and the following theorem is proved.

Theorem 5.1. *The discrete analogue of the differential operator d^{2m}/dx^{2m} has the form*

$$D_m(h\beta) = \frac{(2m-1)!}{h^{2m}} \begin{cases} \sum_{k=1}^{m-1} \frac{(1-q_k)^{2m+1} q_k^{|\beta|}}{q_k E_{2m-1}(q_k)} & \text{for } |\beta| \geq 2, \\ 1 + \sum_{k=1}^{m-1} \frac{(1-q_k)^{2m+1}}{E_{2m-1}(q_k)} & \text{for } |\beta| = 1, \\ -2^{2m-1} + \sum_{k=1}^{m-1} \frac{(1-q_k)^{2m+1}}{q_k E_{2m-1}(q_k)} & \text{for } \beta = 0, \end{cases} \quad (5.6)$$

where $E_{2m-1}(q)$ is the Euler-Frobenius polynomial of degree $2m-1$, q_k are the roots of the Euler-Frobenius polynomial $E_{2m-2}(q)$, $|q_k| < 1$, h is a small positive parameter.

Furthermore several properties of the discrete argument function $D_m(h\beta)$ were proved in [26]. Here we give the following property of the discrete argument function $D_m(h\beta)$ which we need in our computations.

Theorem 5.2. *The discrete argument function $D_m(h\beta)$ and the monomials $(h\beta)^k$ are related to each other as follows*

$$\sum_{\beta=-\infty}^{\infty} D_m(h\beta)(h\beta)^k = \begin{cases} 0 & \text{when } 0 \leq k \leq 2m-1, \\ (2m)! & \text{when } k = 2m, \end{cases} \quad (5.7)$$

$$\sum_{\beta=-\infty}^{\infty} D_m(h\beta)(h\beta)^k = \begin{cases} 0 & \text{when } 2m+1 \leq k \leq 4m-1, \\ \frac{h^{2m}(4m)!B_{2m}}{(2m)!} & \text{when } k = 4m. \end{cases}$$

Then, taking into account (5.5) and Theorems 5.1, 5.2, for the optimal coefficients $C[\beta]$ we have

$$C[\beta] = hD_m(h\beta) * u(h\beta). \quad (5.8)$$

So, if we find the function $u(h\beta)$, then the optimal coefficients $C[\beta]$ will be found from equality (5.8).

To calculate the convolution (5.8) it is required to find the representation of the function $u(h\beta)$ for all integer values of β . From the equality (5.1) we get, that $u(h\beta) = f_m(h\beta)$ when $h\beta \in [0, 1]$, where $f_m(h\beta)$ is defined by equality (5.2). Now we need to find the representation of the function $u(h\beta)$ when $\beta < 0$ and $\beta > N$.

Since $C[\beta] = 0$ when $h\beta \notin [0, 1]$, then

$$C[\beta] = hD_m(h\beta) * u(h\beta) = 0, \quad h\beta \notin [0, 1].$$

Now we calculate the convolution $v(h\beta) = C[\beta] * \frac{|h\beta|^{2m-1}}{2(2m-1)!}$ when $h\beta \notin [0, 1]$.

Suppose $\beta < 0$, then taking into account (4.4)-(4.6), we have

$$\begin{aligned} v(h\beta) &= C[\beta] * \frac{|h\beta|^{2m-1}}{2(2m-1)!} = \sum_{\gamma=-\infty}^{\infty} C[\gamma] \frac{|h\beta - h\gamma|^{2m-1}}{2(2m-1)!} = \\ &= - \sum_{\alpha=0}^{m-1} \frac{(h\beta)^{2m-1-\alpha} (-1)^\alpha}{2(2m-1-\alpha)! \alpha!} \sum_{\gamma=0}^N C[\gamma] (h\gamma)^\alpha - \sum_{\alpha=m}^{2m-1} \frac{(h\beta)^{2m-1-\alpha} (-1)^\alpha}{2(2m-1-\alpha)! \alpha!} \sum_{\gamma=0}^N C[\gamma] (h\gamma)^\alpha = \\ &= - \frac{(h\beta)^{2m-1}}{2(2m-1)!} + \frac{(h\beta)^{2m-2}}{4(2m-2)!} - \frac{(h\beta)^{2m-3}}{4(2m-3)!} \left(\frac{1}{3} + 2A \right) + \frac{(h\beta)^{2m-4}}{2(2m-4)! 3!} \left(\frac{1}{4} + 3A \right) - \\ &\quad - \sum_{\alpha=4}^{m-1} \frac{(h\beta)^{2m-1-\alpha} (-1)^\alpha}{2(2m-1-\alpha)! \alpha!} \left(\frac{1}{\alpha+1} + \alpha A + \alpha(\alpha-1)(\alpha-2)B \right) - \\ &\quad - \sum_{\alpha=m}^{2m-1} \frac{(h\beta)^{2m-1-\alpha} (-1)^\alpha}{2(2m-1-\alpha)! \alpha!} \sum_{\gamma=0}^N C[\gamma] (h\gamma)^\alpha. \end{aligned}$$

Hence, denoting by $R_{m-1}(h\beta) = \sum_{\alpha=m}^{2m-1} \frac{(h\beta)^{2m-1-\alpha} (-1)^\alpha}{2(2m-1-\alpha)! \alpha!} \sum_{\gamma=0}^N C[\gamma] (h\gamma)^\alpha$ for the case $\beta < 0$ we get

$$\begin{aligned} v(h\beta) &= - \frac{(h\beta)^{2m-1}}{2(2m-1)!} + \frac{(h\beta)^{2m-2}}{4(2m-2)!} - \frac{(h\beta)^{2m-3}}{4(2m-3)!} \left(\frac{1}{3} + 2A \right) + \frac{(h\beta)^{2m-4}}{2(2m-4)! 3!} \left(\frac{1}{4} + 3A \right) - \\ &\quad - \sum_{\alpha=4}^{m-1} \frac{(h\beta)^{2m-1-\alpha} (-1)^\alpha}{2(2m-1-\alpha)! \alpha!} \left(\frac{1}{\alpha+1} + \alpha A + \alpha(\alpha-1)(\alpha-2)B \right) - R_{m-1}(h\beta). \end{aligned} \quad (5.9)$$

Now suppose $\beta > N$ then for $v(h\beta)$ we get

$$\begin{aligned} v(h\beta) &= \frac{(h\beta)^{2m-1}}{2(2m-1)!} - \frac{(h\beta)^{2m-2}}{4(2m-2)!} + \frac{(h\beta)^{2m-3}}{4(2m-3)!} \left(\frac{1}{3} + 2A \right) - \frac{(h\beta)^{2m-4}}{2(2m-4)!3!} \left(\frac{1}{4} + 3A \right) + \\ &+ \sum_{\alpha=4}^{m-1} \frac{(h\beta)^{2m-1-\alpha}(-1)^\alpha}{2(2m-1-\alpha)\alpha!} \left(\frac{1}{\alpha+1} + \alpha A + \alpha(\alpha-1)(\alpha-2)B \right) + R_{m-1}(h\beta). \end{aligned} \quad (5.10)$$

Denoting

$$R_{m-1}^{(-)}(h\beta) = \sum_{\alpha=0}^{m-1} \lambda_\alpha (h\beta)^\alpha - R_{m-1}(h\beta), \quad (5.11)$$

$$R_{m-1}^{(+)}(h\beta) = \sum_{\alpha=0}^{m-1} \lambda_\alpha (h\beta)^\alpha + R_{m-1}(h\beta), \quad (5.12)$$

and taking into account (5.9), (5.10), (5.4) we have the following problem

Problem B. Find the solution of the equation

$$hD_m(h\beta) * u(h\beta) = 0, \quad h\beta \notin [0, 1] \quad (5.13)$$

having the form:

$$u(h\beta) = \begin{cases} -\frac{(h\beta)^{2m-1}}{2(2m-1)!} + \frac{(h\beta)^{2m-2}}{4(2m-2)!} - \frac{(h\beta)^{2m-3}}{4(2m-3)!} \left(\frac{1}{3} + 2A \right) + \frac{(h\beta)^{2m-4}}{2(2m-4)!3!} \left(\frac{1}{4} + 3A \right) - \\ - \sum_{\alpha=4}^{m-1} \frac{(h\beta)^{2m-1-\alpha}(-1)^\alpha}{2(2m-1-\alpha)\alpha!} \left(\frac{1}{\alpha+1} + \alpha A + \alpha(\alpha-1)(\alpha-2)B \right) + R_{m-1}^{(-)}(h\beta), & \beta < 0, \\ f(h\beta), & 0 \leq \beta \leq N, \\ \frac{(h\beta)^{2m-1}}{2(2m-1)!} - \frac{(h\beta)^{2m-2}}{4(2m-2)!} + \frac{(h\beta)^{2m-3}}{4(2m-3)!} \left(\frac{1}{3} + 2A \right) - \frac{(h\beta)^{2m-4}}{2(2m-4)!3!} \left(\frac{1}{4} + 3A \right) + \\ + \sum_{\alpha=4}^{m-1} \frac{(h\beta)^{2m-1-\alpha}(-1)^\alpha}{2(2m-1-\alpha)\alpha!} \left(\frac{1}{\alpha+1} + \alpha A + \alpha(\alpha-1)(\alpha-2)B \right) + R_{m-1}^{(+)}(h\beta), & \beta > N, \end{cases} \quad (5.14)$$

where $R_{m-1}^{(-)}(h\beta)$ and $R_{m-1}^{(+)}(h\beta)$ are unknown polynomials of degree $m-1$ and A, B are unknown coefficients.

If we find $R_{m-1}^{(-)}(h\beta)$ and $R_{m-1}^{(+)}(h\beta)$, then from (5.11), (5.12) we obtain

$$\begin{aligned} \sum_{\alpha=0}^{m-1} \lambda_\alpha (h\beta)^\alpha &= \frac{1}{2} \left(R_{m-1}^{(+)}(h\beta) + R_{m-1}^{(-)}(h\beta) \right), \\ R_{m-1}(h\beta) &= \frac{1}{2} \left(R_{m-1}^{(+)}(h\beta) - R_{m-1}^{(-)}(h\beta) \right). \end{aligned}$$

Unknowns $R_{m-1}^{(-)}(h\beta)$, $R_{m-1}^{(+)}(h\beta)$, A and B can be found from equation (5.13), using the discrete argument function $D_m(h\beta)$. Then we can obtain the explicit

form of the function $u(h\beta)$ and respectively we can find the optimal coefficients $C[\beta]$ ($\beta = 0, 1, \dots, N$), A and B . Thus Problem B and respectively Problem A can be solved.

But here we will not find $R_{m-1}^{(-)}(h\beta)$, $R_{m-1}^{(+)}(h\beta)$. Instead, using $D_m(h\beta)$ and the form (5.14) of the discrete argument function $u(h\beta)$, taking into account (5.8), we find the expressions for the optimal coefficients $C[\beta]$ when $\beta = 1, 2, \dots, N - 1$.

We introduce the following notations

$$d_k = \frac{(2m-1)!(1-q_k)^{2m+1}}{h^{2m}q_k E_{2m-1}(q_k)} \sum_{\gamma=1}^{\infty} q_k^{\gamma} \left\{ \frac{(h\gamma)^{2m-1}}{2(2m-1)!} + \frac{(h\gamma)^{2m-2}}{4(2m-2)!} - \frac{(h\gamma)^{2m-3}}{4(2m-3)!} \left(\frac{1}{3} + 2A \right) + \right. \\ \left. + \frac{(h\gamma)^{2m-4}}{12(2m-4)!} \left(\frac{1}{4} + 3A \right) + \sum_{\alpha=4}^{m-1} \frac{(h\gamma)^{2m-1-\alpha}(-1)^{\alpha}}{2(2m-1-\alpha)\alpha!} \left(\frac{1}{\alpha+1} + \alpha A + \alpha(\alpha-1)(\alpha-2)B \right) + \right. \\ \left. + R_{m-1}^{(-)}(-h\gamma) - f(-h\gamma) \right\}, \quad (5.15)$$

$$p_k = \frac{(2m-1)!(1-q_k)^{2m+1}}{h^{2m}q_k E_{2m-1}(q_k)} \sum_{\gamma=1}^{\infty} q_k^{\gamma} \left\{ \frac{(h(\gamma+N))^{2m-1}}{2(2m-1)!} + \frac{(h(\gamma+N))^{2m-2}}{4(2m-2)!} - \right. \\ \left. - \frac{(h(\gamma+N))^{2m-3}}{4(2m-3)!} \left(\frac{1}{3} + 2A \right) + \frac{(h(\gamma+N))^{2m-4}}{12(2m-4)!} \left(\frac{1}{4} + 3A \right) + \right. \\ \left. + \sum_{\alpha=4}^{m-1} \frac{(h(\gamma+N))^{2m-1-\alpha}(-1)^{\alpha}}{2(2m-1-\alpha)\alpha!} \left(\frac{1}{\alpha+1} + \alpha A + \alpha(\alpha-1)(\alpha-2)B \right) + \right. \\ \left. + R_{m-1}^{(+)}(h\gamma+1) - f(h\gamma+1) \right\}, \quad (5.16)$$

where $k = 1, 2, \dots, m-1$, $E_{2m-1}(q)$ is the Euler-Frobenius polynomial of degree $2m-1$, q_k are given in Theorem 5.1. Note that because of $|q_k| < 1$ the series in the (5.15) and (5.16) are convergent.

The following holds

Theorem 5.3. *The coefficients $C[\beta]$, $\beta = 1, 2, \dots, N-1$ of the optimal quadrature formulas of the form (1.4) in the space $L_2^{(m)}(0, 1)$ for $m \geq 4$ have the following form*

$$C[\beta] = h \left(1 + \sum_{k=1}^{m-1} \left(d_k q_k^{\beta} + p_k q_k^{N-\beta} \right) \right), \quad \beta = 1, 2, \dots, N-1, \quad (5.17)$$

where d_k, p_k are defined by (5.15), (5.16), q_k are given in Theorem 5.1.

Proof. Suppose $\beta = 1, 2, \dots, N-1$. Then from (5.8), using Definition 2.3,

equalities (5.6), (5.14), we have

$$\begin{aligned} C[\beta] &= hD_m(h\beta) * u(h\beta) = h \sum_{\gamma=-\infty}^{\infty} D_m(h\beta - h\gamma)u(h\gamma) = \\ &= h \left(\sum_{\gamma=-\infty}^{-1} D_m(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=0}^N D_m(h\beta - h\gamma)f_m(h\gamma) + \sum_{\gamma=N+1}^{\infty} D_m(h\beta - h\gamma)u(h\gamma) \right). \end{aligned}$$

Now, adding and subtracting the expressions $h \sum_{\gamma=-\infty}^{-1} D_m(h\beta - h\gamma)f_m(h\gamma)$ and $h \sum_{\gamma=N+1}^{\infty} D_m(h\beta - h\gamma)f_m(h\gamma)$ to and from the last expression and taking into account Definition 2.3 we get

$$\begin{aligned} C[\beta] &= h \left\{ D_m(h\beta) * f_m(h\beta) + \sum_{k=1}^{m-1} q_k^\beta \frac{(2m-1)!(1-q_k)^{2m+1}}{h^{2m}} \frac{1}{q_k E_{2m-1}(q_k)} \sum_{\gamma=1}^{\infty} q_k^\gamma \left[-\frac{(h\gamma)^{2m-1}}{2(2m-1)!} + \right. \right. \\ &\quad + \frac{(h\gamma)^{2m-2}}{4(2m-2)!} - \frac{(h\gamma)^{2m-3}}{4(2m-3)!} \left(\frac{1}{3} + 2A \right) + \frac{(h\gamma)^{2m-4}}{12(2m-4)!} \left(\frac{1}{4} + 3A \right) - \\ &\quad - \sum_{\alpha=4}^{m-1} \frac{(h\gamma)^{2m-1-\alpha}(-1)^\alpha}{2(2m-1-\alpha)\alpha!} \left(\frac{1}{\alpha+1} + \alpha A + \alpha(\alpha-1)(\alpha-2)B \right) + \\ &\quad \left. + R_{m-1}^{(-)}(-h\gamma) - f_m(-h\gamma) \right] + \\ &\quad + \sum_{k=1}^{m-1} q_k^{N-\beta} \frac{(2m-1)!(1-q_k)^{2m+1}}{h^{2m}} \frac{1}{q_k E_{2m-1}(q_k)} \sum_{\gamma=1}^{\infty} q_k^\gamma \left[\frac{(h(\gamma+N))^{2m-1}}{2(2m-1)!} - \right. \\ &\quad - \frac{(h(\gamma+N))^{2m-2}}{4(2m-2)!} + \frac{(h(\gamma+N))^{2m-3}}{4(2m-3)!} \left(\frac{1}{3} + 2A \right) - \frac{(h(\gamma+N))^{2m-4}}{12(2m-4)!} \left(\frac{1}{4} + 3A \right) + \\ &\quad + \sum_{\alpha=4}^{m-1} \frac{(h(\gamma+N))^{2m-1-\alpha}(-1)^\alpha}{2(2m-1-\alpha)\alpha!} \left(\frac{1}{\alpha+1} + \alpha A + \alpha(\alpha-1)(\alpha-2)B \right) + \\ &\quad \left. \left. + R_{m-1}^{(+)}(h\gamma+1) - f_m(h\gamma+1) \right] \right\}. \end{aligned}$$

Hence taking into account the notations (5.15), (5.16) we obtain

$$C[\beta] = h \left(D_m(h\beta) * f_m(h\beta) + \sum_{k=1}^{m-1} (d_k q_k^\beta + p_k q_k^{N-\beta}) \right). \quad (5.18)$$

Now using Theorems 5.1, 5.2 and equality (5.2) we get

$$\begin{aligned} D_m(h\beta) * f(h\beta) &= D_m(h\beta) * \left(\frac{(h\beta)^{2m}}{(2m)!} + \sum_{j=0}^{2m-1} \frac{(-h\beta)^{2m-1-j}}{2(2m-1-j)!(j+1)!} \right) \\ &= D_m(h\beta) * \frac{(h\beta)^{2m}}{(2m)!} = 1. \end{aligned} \quad (5.19)$$

Putting (5.19) to the equation (5.18) we get (5.17). Theorem 5.3 is proved.

Furthermore we need the following lemmas in the proof of the main results.

Lemma 5.1. *The following identity is taken place*

$$\sum_{i=0}^{\alpha} \frac{d_k q_k + p_k q_k^{N+i} (-1)^{i+1}}{(q_k - 1)^{i+1}} \Delta^i 0^\alpha = (-1)^{\alpha+1} \sum_{i=0}^{\alpha} \frac{d_k q_k^i + p_k q_k^{N+1} (-1)^{i+1}}{(1 - q_k)^{i+1}} \Delta^i 0^\alpha, \quad (5.20)$$

here α and N are natural numbers, d_k and p_k are defined by (5.15), (5.16), $\Delta^i 0^\alpha$ is given in Theorem 2.1, q_k are given in Theorem 5.1.

Proof. For the purposes of convenience the left and the right hand sides of (5.20) we denote by L_1 and $(-1)^{\alpha+1} L_2$ respectively, i.e.

$$L_1 = \sum_{i=0}^{\alpha} \frac{d_k q_k + p_k q_k^{N+i} (-1)^{i+1}}{(q_k - 1)^{i+1}} \Delta^i 0^\alpha \text{ and } L_2 = \sum_{i=0}^{\alpha} \frac{d_k q_k^i + p_k q_k^{N+1} (-1)^{i+1}}{(1 - q_k)^{i+1}} \Delta^i 0^\alpha.$$

First consider L_1 . Using the equality (2.3) and the identity (2.2) for L_1 consequently we get

$$\begin{aligned} L_1 &= \sum_{i=0}^{\alpha} \frac{d_k q_k + p_k q_k^{N+i} (-1)^{i+1}}{(q_k - 1)^{i+1}} \Delta^i 0^\alpha = \frac{d_k q_k}{(q_k - 1)^{\alpha+1}} E_{\alpha-1}(q_k) + \frac{p_k q_k^{N+\alpha} (-1)^{\alpha+1}}{(q_k - 1)^{\alpha+1}} E_{\alpha-1} \left(\frac{1}{q_k} \right) = \\ &= \frac{d_k q_k}{(q_k - 1)^{\alpha+1}} E_{\alpha-1}(q_k) + \frac{p_k q_k^{N+\alpha} (-1)^{\alpha+1}}{(q_k - 1)^{\alpha+1}} \frac{E_{\alpha-1}(q_k)}{q_k^{\alpha-1}} = \frac{d_k q_k + p_k q_k^{N+1} (-1)^{\alpha+1}}{(q_k - 1)^{\alpha+1}} E_{\alpha-1}(q_k). \end{aligned} \quad (5.21)$$

Similarly for L_2 using (2.3) and (2.2) we have

$$\begin{aligned} L_2 &= \sum_{i=0}^{\alpha} \frac{d_k q_k^i + p_k q_k^{N+1} (-1)^{i+1}}{(1 - q_k)^{i+1}} \Delta^i 0^\alpha = \frac{d_k q_k^\alpha}{(q_k - 1)^{\alpha+1}} E_{\alpha-1} \left(\frac{1}{q_k} \right) + \frac{p_k q_k^{N+1}}{(q_k - 1)^{\alpha+1}} E_{\alpha-1}(q_k) = \\ &= \frac{d_k q_k}{(1 - q_k)^{\alpha+1}} E_{\alpha-1}(q_k) + \frac{p_k q_k^{N+1}}{(q_k - 1)^{\alpha+1}} E_{\alpha-1}(q_k) = \frac{d_k q_k (-1)^{\alpha+1} + p_k q_k^{N+1}}{(q_k - 1)^{\alpha+1}} E_{\alpha-1}(q_k) = \\ &= (-1)^{\alpha+1} \frac{d_k q_k + p_k q_k^{N+1} (-1)^{\alpha+1}}{(q_k - 1)^{\alpha+1}} E_{\alpha-1}(q_k). \end{aligned} \quad (5.22)$$

From (5.21) and (5.22) it is clear, that $L_1 = (-1)^{\alpha+1} L_2$. Lemma 5.1 is proved.

We denote

$$Z_p = \sum_{k=1}^{m-1} \sum_{i=0}^p \frac{d_k q_k^{N+i} + p_k q_k (-1)^{i+1}}{(1 - q_k)^{i+1}} \Delta^i 0^p. \quad (5.23)$$

Lemma 5.2. *The following identities are valid*

$$\begin{aligned} \sum_{j=0}^{m-1} \frac{(-1)^{j+1}}{(j+1)!} \sum_{i=1}^{2m-2-j} \frac{B_{2m-j-i} h^{2m-j-i}}{i! (2m-j-i)!} &= \sum_{j=2}^m \frac{B_j h^j}{j!} \sum_{i=1}^m \frac{(-1)^i}{i! (2m+1-j-i)!} + \\ &+ \sum_{j=m+1}^{2m-1} \frac{B_j h^j}{j!} \sum_{i=1}^{2m-j} \frac{(-1)^i}{i! (2m+1-j-i)!} \end{aligned}$$

and

$$\begin{aligned} \sum_{j=0}^{m-1} \frac{(-1)^{j+1}}{(j+1)!} \sum_{p=1}^{2m-1-j} \frac{h^{p+1} Z_p}{p! (2m-1-j-p)!} &= \sum_{j=2}^{m+1} \frac{h^j Z_{j-1}}{(j-1)!} \sum_{i=1}^m \frac{(-1)^i}{i! (2m+1-j-i)!} + \\ &\sum_{j=m+2}^{2m} \frac{h^j Z_{j-1}}{(j-1)!} \sum_{i=1}^{2m+1-j} \frac{(-1)^i}{i! (2m+1-j-i)!}. \end{aligned}$$

The proof of Lemma 5.2 is obtained by expansion in powers of h of the left sides of given identities.

For the coefficients of the optimal quadrature formulas of the form (1.4) the following theorem holds.

Theorem 5.4. *Among quadrature formulas of the form (1.4) with the error functional (1.5) in the space $L_2^{(m)}(0, 1)$ for $m \geq 4$ there exists unique optimal formula which coefficients are determined by the following formulas*

$$C[0] = h \left(\frac{1}{2} + \sum_{k=1}^{m-1} d_k \frac{q_k^N - q_k}{1 - q_k} \right), \quad (5.24)$$

$$C[\beta] = h \left(1 + \sum_{k=1}^{m-1} d_k (q_k^\beta + q_k^{N-\beta}) \right), \quad \beta = \overline{1, N-1}, \quad (5.25)$$

$$C[N] = h \left(\frac{1}{2} + \sum_{k=1}^{m-1} d_k \frac{q_k^N - q_k}{1 - q_k} \right), \quad (5.26)$$

$$A = h^2 \left(\frac{1}{12} - \sum_{k=1}^{m-1} d_k \frac{q_k + q_k^{N+1}}{(1 - q_k)^2} \right), \quad (5.27)$$

$$B = h^4 \left(\frac{B_4}{4!} - \frac{1}{3!} \sum_{k=1}^{m-1} d_k \sum_{i=0}^3 \frac{q_k + (-1)^{i+1} q_k^{N+i}}{(q_k - 1)^{i+1}} \Delta^i 0^3 \right), \quad (5.28)$$

where d_k satisfy the following system of $m-1$ linear equations

$$\sum_{k=1}^{m-1} d_k \sum_{i=0}^j \frac{q_k + (-1)^{i+1} q_k^{N+i}}{(q_k - 1)^{i+1}} \Delta^i 0^j = \frac{B_{j+1}}{j+1}, \quad j = \overline{4, m-1}, \quad (5.29)$$

$$\sum_{k=1}^{m-1} d_k \sum_{i=0}^j \frac{q_k + (-1)^{i+1} q_k^{N+i}}{(q_k - 1)^{i+1}} \Delta^i 0^j = 0, \quad j = 2, 2m-4, 2m-2. \quad (5.30)$$

Here B_α are Bernoulli numbers, $\Delta^i \gamma^j$ is the finite difference of order i of γ^j , $\Delta^i 0^j$ is given in Theorem 2.1, q_k are given in Theorem 5.1.

Proof. First we consider the first sum of equation (4.1). For this sum we have

$$\begin{aligned} S &= \sum_{\gamma=0}^N C[\gamma] \frac{|h\beta - h\gamma|^{2m-1}}{2(2m-1)!} = \\ &= C[0] \frac{(h\beta)^{2m-1}}{(2m-1)!} + \sum_{\gamma=1}^{\beta} C[\gamma] \frac{(h\beta - h\gamma)^{2m-1}}{(2m-1)!} - \sum_{\gamma=0}^N C[\gamma] \frac{(h\beta - h\gamma)^{2m-1}}{2(2m-1)!}. \end{aligned}$$

The last two sums of the expression S we denote

$$S_1 = \sum_{\gamma=1}^{\beta} C[\gamma] \frac{(h\beta - h\gamma)^{2m-1}}{(2m-1)!}, \quad S_2 = \sum_{\gamma=0}^N C[\gamma] \frac{(h\beta - h\gamma)^{2m-1}}{2(2m-1)!}$$

and we calculate them separately.

By using (5.17) and formulas (2.4), (2.5) for S_1 we have

$$\begin{aligned} S_1 &= \sum_{\gamma=0}^{\beta} h \left(1 + \sum_{k=1}^{m-1} (d_k q_k^{\gamma} + p_k q_k^{N-\gamma}) \right) \frac{(h\beta - h\gamma)^{2m-1}}{(2m-1)!} = \\ &= \frac{h^{2m}}{(2m-1)!} \left[\sum_{\gamma=0}^{\beta-1} \gamma^{2m-1} + \sum_{k=1}^{m-1} \left(d_k q_k^{\beta} \sum_{\gamma=0}^{\beta-1} q_k^{-\gamma} \gamma^{2m-1} + p_k q_k^{N-\beta} \sum_{\gamma=0}^{\beta-1} q_k^{\gamma} \gamma^{2m-1} \right) \right] = \\ &= \frac{h^{2m}}{(2m-1)!} \left[\sum_{j=1}^{2m} \frac{(2m-1)! B_{2m-j}}{j! \cdot (2m-j)!} \beta^j + \sum_{k=1}^{m-1} \left[d_k q_k^{\beta} \left\{ \frac{q_k}{q_k-1} \sum_{i=0}^{2m-1} \frac{\Delta^i 0^{2m-1}}{(q_k-1)^i} - \right. \right. \right. \\ &\quad \left. \left. - \frac{q_k^{1-\beta}}{q_k-1} \sum_{i=0}^{2m-1} \frac{\Delta^i \beta^{2m-1}}{(q_k-1)^i} \right\} + p_k q_k^{N-\beta} \left\{ \frac{1}{1-q_k} \sum_{i=0}^{2m-1} \left(\frac{q_k}{q_k-1} \right)^i \Delta^i 0^{2m-1} - \right. \right. \\ &\quad \left. \left. - \frac{q_k^{\beta}}{1-q_k} \sum_{i=0}^{2m-1} \left(\frac{q_k}{q_k-1} \right)^i \Delta^i \beta^{2m-1} \right\} \right] \right]. \end{aligned}$$

Taking into account that q_k is the root of the Euler-Frobenius polynomial $E_{2m-2}(q)$ and using formulas (2.3), (2.6) the expression for S_1 we reduce to the following form

$$\begin{aligned} S_1 &= \frac{(h\beta)^{2m}}{(2m)!} + h \cdot \frac{(h\beta)^{2m-1}}{(2m-1)!} B_1 + h^{2m} \sum_{j=1}^{2m-2} \frac{B_{2m-j}}{j!(2m-j)!} \beta^j + \\ &+ h^{2m} \sum_{j=0}^{2m-1} \frac{\beta^{2m-1-j}}{j!(2m-1-j)!} \sum_{k=1}^{m-1} \sum_{i=0}^j \frac{-d_k q_k + p_k q_k^{N+i} (-1)^i}{(q_k-1)^{i+1}} \Delta^i 0^j. \end{aligned} \quad (5.31)$$

Now we consider S_2 . By using equations (4.4)-(4.6) we rewrite the expression S_2 in powers of $h\beta$

$$\begin{aligned}
S_2 &= \sum_{\gamma=0}^N C[\gamma] \frac{(h\beta - h\gamma)^{2m-1}}{2(2m-1)!} = \\
&= \sum_{j=0}^{m-1} \frac{(h\beta)^{2m-1-j} (-1)^j}{2(2m-1-j)! j!} \sum_{\gamma=0}^N C[\gamma] (h\gamma)^j + \sum_{j=m}^{2m-1} \frac{(h\beta)^{2m-1-j} (-1)^j}{2(2m-1-j)! j!} \sum_{\gamma=0}^N C[\gamma] (h\gamma)^j = \\
&= \frac{(h\beta)^{2m-1}}{2(2m-1)!} - \frac{(h\beta)^{2m-2}}{4(2m-2)!} + \frac{(h\beta)^{2m-3}}{4(2m-3)!} \left(\frac{1}{3} + 2A \right) - \frac{(h\beta)^{2m-4}}{2(2m-4)! 3!} \left(\frac{1}{4} + 3A \right) + \\
&\quad + \sum_{j=4}^{m-1} \frac{(h\beta)^{2m-1-j} (-1)^j}{2(2m-1-j)! j!} \left(\frac{1}{j+1} + jA + j(j-1)(j-2)B \right) + \\
&\quad + \sum_{j=m}^{2m-1} \frac{(h\beta)^{2m-1-j} (-1)^j}{2(2m-1-j)! j!} \sum_{\gamma=0}^N C[\gamma] (h\gamma)^j. \tag{5.32}
\end{aligned}$$

Substituting (5.2) and S into equation (4.1) and using (5.31), (5.32) we have

$$\begin{aligned}
&\frac{(h\beta)^{2m}}{(2m)!} + C[0] \frac{(h\beta)^{2m-1}}{(2m-1)!} + h \frac{(h\beta)^{2m-1}}{(2m-1)!} B_1 + \sum_{j=1}^{2m-2} \frac{B_{2m-j} h^{2m-j} (h\beta)^j}{j! (2m-j)!} + \\
&\quad + \sum_{j=0}^{2m-1} \frac{h^{j+1} (h\beta)^{2m-1-j}}{j! (2m-1-j)!} \sum_{k=1}^{m-1} \sum_{i=0}^j \frac{-d_k q_k + p_k q_k^{N+i} (-1)^i}{(q_k - 1)^{i+1}} \Delta^i 0^j - \\
&\quad - \frac{(h\beta)^{2m-1}}{2(2m-1)!} + \frac{(h\beta)^{2m-2}}{4(2m-2)!} - \frac{(h\beta)^{2m-3}}{4(2m-3)!} \left(\frac{1}{3} + 2A \right) + \frac{(h\beta)^{2m-4}}{2(2m-4)! 3!} \left(\frac{1}{4} + 3A \right) - \\
&\quad - \sum_{j=4}^{m-1} \frac{(h\beta)^{2m-1-j} (-1)^j}{2(2m-1-j)! j!} \left(\frac{1}{j+1} + jA + j(j-1)(j-2)B \right) - \\
&\quad - \sum_{j=m}^{2m-1} \frac{(h\beta)^{2m-1-j} (-1)^j}{2(2m-1-j)! j!} \sum_{\gamma=0}^N C[\gamma] (h\gamma)^j \\
&\quad - A \frac{(h\beta)^{2m-2}}{2(2m-2)!} - A \sum_{j=0}^{2m-2} \frac{(h\beta)^{2m-2-j} (-1)^j}{2 \cdot j! \cdot (2m-2-j)!} - B \frac{(h\beta)^{2m-4}}{2(2m-4)!} - \\
&\quad - B \sum_{j=0}^{2m-4} \frac{(h\beta)^{2m-4-j} (-1)^j}{2 \cdot j! \cdot (2m-4-j)!} + \sum_{\alpha=0}^{m-1} \lambda_\alpha (h\beta)^\alpha = \frac{(h\beta)^{2m}}{(2m)!} + \sum_{j=0}^{2m-1} \frac{(-h\beta)^{2m-1-j}}{2 \cdot (2m-1-j)! \cdot (j+1)!}.
\end{aligned}$$

Hence equating coefficients of the same powers of $h\beta$ gives

$$\begin{aligned}
\sum_{j=0}^{m-1} \lambda_j (h\beta)^j &= \sum_{j=0}^{m-1} \frac{(h\beta)^j}{j!} \left[\frac{(-1)^j}{2(2m-j)!} + \frac{B(-1)^j}{2(2m-4-j)!} + \frac{A(-1)^j}{2(2m-2-j)!} - \right. \\
&\quad \left. - \frac{h^{2m-j}}{(2m-1-j)!} \sum_{k=1}^{m-1} \sum_{i=0}^{2m-1-j} \frac{-d_k q_k + p_k q_k^{N+i} (-1)^i}{(q_k - 1)^{i+1}} \Delta^i 0^{2m-1-j} + \right.
\end{aligned}$$

$$+ \frac{1}{2(2m-1-j)!} \sum_{\gamma=0}^N C[\gamma] (-h\gamma)^{2m-1-j} \left] - \sum_{j=1}^{m-1} \frac{B_{2m-j} h^{2m-j} (h\beta)^j}{j!(2m-j)!}, \quad (5.33)$$

$$\sum_{k=1}^{m-1} \sum_{i=0}^{j+4} \frac{d_k q_k + p_k q_k^{N+i} (-1)^{i+1}}{(q_k - 1)^{i+1}} \Delta^i 0^{j+4} = \frac{B_{j+5}}{j+5}, \quad j = \overline{0, m-5}, \quad (5.34)$$

$$\sum_{k=1}^{m-1} d_k \sum_{i=0}^2 \frac{q_k^{N+i} (-1)^{i+1} + q_k}{(q_k - 1)^{i+1}} \Delta^i 0^2 = 0, \quad (5.35)$$

$$C[0] = h \left(\frac{1}{2} + \sum_{k=1}^{m-1} \frac{p_k q_k^N - d_k q_k}{1 - q_k} \right), \quad (5.36)$$

$$A = h^2 \left(\frac{1}{12} - \sum_{k=1}^{m-1} \frac{d_k q_k + p_k q_k^{N+1}}{(1 - q_k)^2} \right), \quad (5.37)$$

$$B = h^4 \left(-\frac{1}{720} + \frac{1}{3!} \sum_{k=1}^{m-1} \sum_{i=0}^3 \frac{d_k q_k + p_k q_k^{N+i} (-1)^{i+1}}{(q_k - 1)^{i+1}} \Delta^i 0^3 \right). \quad (5.38)$$

Substituting the expressions (5.36) into (4.4), also taking into account (5.17), we find $C[N]$ which have the following form

$$C[N] = h \left(\frac{1}{2} + \sum_{k=1}^{m-1} \frac{d_k q_k^N - p_k q_k}{1 - q_k} \right). \quad (5.39)$$

Now, putting the values of λ_j into (4.2) the following equations for we get unknowns d_k and p_k :

$$\sum_{k=1}^{m-1} \sum_{i=0}^{2m-2} \frac{d_k q_k^i + p_k q_k^{N+1} (-1)^{i+1}}{(1 - q_k)^{i+1}} \Delta^i 0^{2m-2} = \sum_{k=1}^{m-1} \sum_{i=0}^{2m-2} \frac{d_k q_k^{N+i} + p_k q_k (-1)^{i+1}}{(1 - q_k)^{i+1}} \Delta^i 0^{2m-2}, \quad (5.40)$$

$$\sum_{k=1}^{m-1} \sum_{i=0}^{j-1} \frac{d_k q_k^{N+i} + p_k q_k (-1)^{i+1}}{(1 - q_k)^{i+1}} \Delta^i 0^{j-1} = \frac{B_j}{j}, \quad (5.41)$$

$$\sum_{k=1}^{m-1} \sum_{i=0}^3 \frac{d_k q_k + p_k q_k^{N+i} (-1)^{i+1}}{(q_k - 1)^{i+1}} \Delta^i 0^3 = \sum_{k=1}^{m-1} \sum_{i=0}^3 \frac{d_k q_k^{N+i} + p_k q_k (-1)^{i+1}}{(1 - q_k)^{i+1}} \Delta^i 0^3, \quad (5.42)$$

$$\sum_{k=1}^{m-1} \sum_{i=0}^2 \frac{d_k q_k^{N+i} + p_k q_k (-1)^{i+1}}{(1 - q_k)^{i+1}} \Delta^i 0^2 = 0, \quad (5.43)$$

$$\sum_{k=1}^{m-1} \frac{d_k q_k + p_k q_k^{N+1}}{(q_k - 1)^2} = \sum_{k=1}^{m-1} \frac{d_k q_k^{N+1} + p_k q_k}{(1 - q_k)^2}, \quad (5.44)$$

$$\sum_{k=1}^{m-1} \sum_{i=0}^{2m-4} \frac{d_k q_k^i + p_k q_k^{N+1} (-1)^{i+1}}{(1 - q_k)^{i+1}} \Delta^i 0^{2m-4} = \sum_{k=1}^{m-1} \sum_{i=0}^{2m-4} \frac{d_k q_k^{N+i} + p_k q_k (-1)^{i+1}}{(1 - q_k)^{i+1}} \Delta^i 0^{2m-4}, \quad (5.45)$$

thus, from equalitions (5.34)- (5.35) and (5.40)- (5.45) we get

$$\sum_{k=1}^{m-1} (d_k - p_k) \sum_{i=0}^{2m-2} \frac{q_k + q_k^{N+i} (-1)^{i+1}}{(q_k - 1)^{i+1}} \Delta^i 0^{2m-2} = 0, \quad (5.46)$$

$$\sum_{k=1}^{m-1} (d_k - p_k) \sum_{i=0}^{2m-4} \frac{q_k + q_k^{N+i} (-1)^{i+1}}{(q_k - 1)^{i+1}} \Delta^i 0^{2m-4} = 0, \quad (5.47)$$

$$\sum_{k=1}^{m-1} (d_k - p_k) \sum_{i=0}^2 \frac{q_k + q_k^{N+i} (-1)^{i+1}}{(q_k - 1)^{i+1}} \Delta^i 0^2 = 0, \quad (5.48)$$

$$\sum_{k=1}^{m-1} (d_k - p_k) \sum_{i=0}^j \frac{q_k + q_k^{N+i} (-1)^{i+1}}{(q_k - 1)^{i+1}} \Delta^i 0^j = \frac{B_{j+1}}{j+1}, \quad j = \overline{4, m-1}, \quad (5.49)$$

Taking into account uniqueness of the optimal coefficients, we conclude, that the homogeneous system of linear equations (5.46)-(5.49) has trivial solution. This means, that

$$d_k = p_k, \quad k = 1, 2, \dots, m-1. \quad (5.50)$$

Then, using (5.50), from (5.46)- (5.49) we get (5.29), (5.30), and from (5.17), (5.36)-(5.39) we obtain (5.24)-(5.28).

Theorem 5.4 is proved.

5.2 The norm of the error functional of optimal quadrature formulas of the form (1.4)

For square of the norm of the error functional (1.5) of optimal quadrature formulas of the form (1.4) the following holds

Theorem 5.5. *For square of the norm of the error functional (1.5) of the optimal quadrature formula of the form (1.4) on the space $L_2^{(m)}(0, 1)$ for $m \geq 4$ the following holds*

$$\left\| \overset{\circ}{\ell} | L_2^{(m)*}(0, 1) \right\|^2 = (-1)^{m+1} \left[\frac{B_{2m} h^{2m}}{(2m)!} - \frac{2h^{2m+1}}{(2m)!} \sum_{k=1}^{m-1} d_k \sum_{i=0}^{2m} \frac{q_k + q_k^{N+i} (-1)^{i+1}}{(q_k - 1)^{i+1}} \Delta^i 0^{2m} \right],$$

where d_k are determined from the system (5.29)-(5.30), B_{2m} are Bernoulli numbers, $\Delta^i 0^{2m}$ is given in Theorem 2.1, q_k are given in Theorem 5.1.

Proof. Computing definite integrals in the expression (3.5) of $\|\ell\|^2$ we get

$$\begin{aligned} \|\ell\|^2 = & (-1)^m \left[\sum_{\beta=0}^N C[\beta] \left(\sum_{\gamma=0}^N C[\gamma] \frac{|h\beta - h\gamma|^{2m-1}}{2(2m-1)!} - \right. \right. \\ & \left. \left. - 2 \int_0^1 \frac{|x - h\beta|^{2m-1}}{2(2m-1)!} dx - 2AP_{2m-2}(h\beta) - 2BP_{2m-4}(h\beta) \right) + \right. \\ & \left. + \frac{2A}{(2m-1)!} + \frac{2B + A^2}{(2m-3)!} + \frac{2AB}{(2m-5)!} + \frac{B^2}{(2m-7)!} + \frac{1}{(2m+1)!} \right]. \end{aligned}$$

where $f(h\beta)$ is defined by formula (5.2). As is obvious from here according to (4.1) the expression into curly brackets is equal to the polynomial $\sum_{\alpha=0}^{m-1} \lambda_{\alpha}(h\beta)^{\alpha}$. Then $\|\ell\|^2$ has the form

$$\begin{aligned} \|\ell\|^2 = & (-1)^m \left[\sum_{\beta=0}^N C[\beta] \left(- \sum_{j=0}^{m-1} \lambda_j(h\beta)^j - f(h\beta) \right) - \right. \\ & - A \sum_{\beta=0}^N C[\beta] P_{2m-2}(h\beta) - B \sum_{\beta=0}^N C[\beta] P_{2m-4}(h\beta) + \\ & \left. + \frac{2A}{(2m-1)!} + \frac{2B+A^2}{(2m-3)!} + \frac{2AB}{(2m-5)!} + \frac{B^2}{(2m-7)!} + \frac{1}{(2m+1)!} \right]. \end{aligned}$$

Hence using (4.2) and (4.3) we get

$$\begin{aligned} \|\ell\|^2 = & (-1)^m \left[- \sum_{j=0}^{m-1} \lambda_j \sum_{\beta=0}^N C[\beta] (h\beta)^j - \sum_{\beta=0}^N C[\beta] f(h\beta) + \right. \\ & + A \sum_{j=2}^{m-1} j \lambda_j + B \sum_{j=4}^{m-1} j(j-1)(j-2) \lambda_j + \\ & \left. + \frac{A}{(2m-1)!} + \frac{B}{(2m-3)!} + \frac{1}{(2m+1)!} \right]. \end{aligned} \quad (5.51)$$

From (5.51) after some simplifications, using (5.33), (5.2) and (4.4)-(4.6) we have

$$\begin{aligned} \|\ell\|^2 = & (-1)^m \left[- \sum_{j=0}^{m-1} \frac{(-1)^{j+1}}{(j+1)!(2m-1-j)!} \sum_{\gamma=0}^N C[\gamma] (h\gamma)^{2m-1-j} - \right. \\ & - \sum_{j=0}^{m-1} \frac{h^{2m-j}}{(j+1)!(2m-1-j)!} \sum_{k=1}^{m-1} d_k \sum_{i=0}^{2m-1-j} \frac{q_k + q_k^{N+i} (-1)^{i+1}}{(q_k-1)^{i+1}} \Delta^i 0^{2m-1-j} \\ & - \sum_{j=0}^{m-1} \frac{(-1)^j}{(j+1)!(2m-j)!} - \sum_{j=0}^{m-5} \frac{B(-1)^j}{(j+1)!(2m-4-j)!} - \\ & - \sum_{j=0}^{m-3} \frac{A(-1)^j}{(j+1)!(2m-2-j)!} + \sum_{j=1}^{m-1} \frac{B_{2m-j} h^{2m-j}}{(j+1)!(2m-j)!} \\ & \left. - \frac{1}{(2m)!} \sum_{\beta=0}^N C[\beta] (h\beta)^{2m} + \frac{A}{(2m-1)!} + \frac{B}{(2m-3)!} + \frac{1}{(2m+1)!} \right] \end{aligned} \quad (5.52)$$

When $\alpha > m-1$ using (5.17) and formulas (2.4)-(2.6) we get

$$\sum_{\gamma=0}^N C[\gamma] (h\gamma)^{\alpha} = \frac{1}{\alpha+1} + \sum_{j=1}^{\alpha-1} \frac{\alpha! B_{\alpha+1-j}}{j!(\alpha+1-j)!} h^{\alpha+1-j} +$$

$$\begin{aligned}
& + h^{\alpha+1} \sum_{k=1}^{m-1} \sum_{i=0}^{\alpha} \frac{d_k q_k^i + p_k q_k^{N+1} (-1)^{i+1}}{(1 - q_k)^{i+1}} \Delta^i 0^\alpha - \\
& - \sum_{j=1}^{\alpha} \frac{\alpha! h^{j+1}}{j! (\alpha - j)!} \sum_{k=1}^{m-1} \sum_{i=0}^j \frac{d_k q_k^{N+i} + p_k q_k (-1)^{i+1}}{(1 - q_k)^{i+1}} \Delta^i 0^j. \tag{5.53}
\end{aligned}$$

Using Lemmas 5.1, 5.2 and taking into account (2.1), (2.3), after some simplifications, from (5.52) we have

$$\begin{aligned}
\|\ell\|^2 = & (-1)^m \left[K_1 + K_2 - \frac{B_{2m} h^{2m}}{(2m)!} + \frac{h^{2m+1} Z_{2m}}{(2m)!} \right. \\
& - \frac{h^{2m+1}}{(2m)!} \sum_{k=1}^{m-1} d_k \sum_{i=0}^{2m} \frac{q_k^i + q_k^{N+1} (-1)^{i+1}}{(1 - q_k)^{i+1}} \Delta^i 0^{2m} - \\
& - \sum_{j=2}^m \frac{B_j h^j}{j!} \sum_{i=0}^m \frac{(-1)^i}{i! (2m+1-i-j)!} + \sum_{j=2}^m \frac{h^j Z_{j-1}}{(j-1)!} \sum_{i=0}^m \frac{(-1)^i}{i! (2m+1-i-j)!} \\
& \left. + \sum_{j=0}^{m-4} \frac{B(-1)^j}{j! (2m-3-j)!} + \sum_{j=0}^{m-2} \frac{A(-1)^j}{j! (2m-1-j)!} \right] \tag{5.54}
\end{aligned}$$

where

$$\begin{aligned}
K_1 = & \sum_{j=m+1}^{2m} \frac{h^j Z_{j-1}}{(j-1)!} \sum_{i=1}^{2m+1-j} \frac{(-1)^i}{i! (2m+1-i-j)!} + \sum_{j=m+1}^{2m} \frac{h^j Z_{j-1}}{(j-1)! (2m+1-j)!} = \\
= & \sum_{j=m+1}^{2m} \frac{h^j Z_{j-1}}{(j-1)!} \sum_{i=0}^{2m+1-j} \frac{(-1)^i}{i! (2m+1-i-j)!} = \sum_{j=m+1}^{2m} \frac{h^j Z_{j-1}}{(j-1)!} \frac{(1-1)^{2m+1-j}}{(2m+1-j)!} = 0,
\end{aligned}$$

and

$$\begin{aligned}
K_2 = & - \sum_{j=m+1}^{2m-1} \frac{h^j B_j}{j!} \sum_{i=0}^{2m-j} \frac{(-1)^i}{i! (2m+1-i-j)!} + \sum_{j=m+1}^{2m-1} \frac{h^j B_j}{j! (2m+1-j)!} = \\
& - \sum_{j=m+1}^{2m-1} \frac{h^j B_j}{j!} \left(\sum_{i=0}^{2m-j} \frac{(-1)^i}{i! (2m+1-i-j)!} - \frac{1}{(2m+1-j)!} \right) = \\
& = - \sum_{j=m+1}^{2m-1} \frac{h^j B_j}{j!} \left(\frac{(-1)^j - 1}{(2m+1-j)!} \right) = 0
\end{aligned}$$

Since $K_1 = K_2 = 0$ then from (5.54) we get the following

$$\begin{aligned}
\|\ell\|^2 = & (-1)^m \left[-\frac{B_{2m} h^{2m}}{(2m)!} + \frac{h^{2m+1}}{(2m)!} \sum_{k=1}^{m-1} d_k \sum_{i=0}^{2m} \frac{q_k^i + q_k^{N+1} (-1)^{i+1}}{(q_k - 1)^{i+1}} \Delta^i 0^{2m} \right. \\
& \left. - \frac{h^{2m+1}}{(2m)!} \sum_{k=1}^{m-1} d_k \sum_{i=0}^{2m} \frac{q_k^i + q_k^{N+1} (-1)^{i+1}}{(1 - q_k)^{i+1}} \Delta^i 0^{2m} \right]
\end{aligned}$$

Hance using Lemma 5.1 we have the statimant of the theorem Theorem 5.5 is proved.

Corollary 5.1. *In the space $L_2^{(4)}(0, 1)$ among quadrature formulas of the form (1.4) with the error functional (1.5) there exists unique optimal formula whose coefficients are determined by the following formulas*

$$C[\beta] = \begin{cases} \frac{h}{2}, & \beta = 0, N, \\ h, & \beta = \overline{1, N-1}, \end{cases}$$

$$A = \frac{h^2}{12}, \quad B = -\frac{h^4}{720}.$$

Furthermore for square of the norm of the error functional the following is valid

$$\left\| \overset{\circ}{\ell} | L_2^{(4)*}(0, 1) \right\|^2 = \frac{h^8}{1209600}.$$

Corollary 5.2. *In the space $L_2^{(5)}(0, 1)$ among quadrature formulas of the form (1.4) with the error functional (1.5) there exists unique optimal formula whose coefficients are determined by the following formulas*

$$C[\beta] = \begin{cases} \frac{h}{2}, & \beta = 0, N, \\ h, & \beta = \overline{1, N-1}, \end{cases}$$

$$A = \frac{h^2}{12}, \quad B = -\frac{h^4}{720}.$$

Furthermore for square of the norm of the error functional the following is valid

$$\left\| \overset{\circ}{\ell} | L_2^{(5)*}(0, 1) \right\|^2 = \frac{h^{10}}{47900160}.$$

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References

- [1] I. Babuška, Optimal quadrature formulas, Dokladi Akad. Nauk SSSR. 149 (1963) 227–229. (in Russian).

- [2] P. Blaga, Gh. Coman, Some problems on optimal quadrature, Stud. Univ. Babeş-Bolyai Math. 52, no. 4 (2007) 21–44.
- [3] B. Bojanov, Optimal quadrature formulas, Uspekhi Mat. Nauk. 60, no. 6(366) (2005) 33–52. (in Russian)
- [4] T. Catinaş, Gh. Coman, Optimal quadrature formulas based on the ϕ -function method, Stud. Univ. Babeş-Bolyai Math. 51, no. 1 (2006) 49–64.
- [5] M.A. Chakhkiev, Linear differential operators with real spectrum, and optimal quadrature formulas, Izv. Akad. Nauk SSSR Ser. Mat. 48, no. 5 (1984) 1078–1108. (in Russian)
- [6] Gh. Coman, Quadrature formulas of Sard type (Romanian), Studia Univ. Babeş-Bolyai Ser. Math.-Mech. 17, no. 2 (1972) 73–77.
- [7] Gh. Coman, Monosplines and optimal quadrature formulae in L_p , Rend. Mat. (6) 5 (1972) 567–577.
- [8] A.O. Gelfond, Calculus of Finite Differences, Nauka, Moscow, 1967. (in Russian)
- [9] A. Ghizzetti, A. Ossicini, Quadrature Formulae, Akademie Verlag, Berlin, 1970.
- [10] R.W. Hamming, Numerical methods for Scientists and Engineers, McGraw Bill Book Company, Inc., USA, 1962.
- [11] A.R. Hayotov, G.V. Milovanović, Kh.M. Shadimetov, On an optimal quadrature formula in the sense of Sard. Numerical Algorithms, v.57, no. 4, (2011) 487–510.
- [12] A.R. Hayotov, F.A. Nuraliev, Kh.M. Shadimetov, Optimal Quadrature Formulas with Derivative in the Space $L_2^{(m)}(0, 1)$. American Journal of Numerical Analysis, 2014, Vol. 2, No. 4, 115–127.
- [13] P. Köhler, On the weights of Sard’s quadrature formulas, Calcolo, 25 (1988) 169–186.
- [14] F. Lanzara, On optimal quadrature formulae, J. Ineq. Appl. 5 (2000) 201–225.
- [15] A.A. Maljukov, I.I. Orlov, Construction of coefficients of the best quadrature formula for the class $W_{L_2}^{(2)}(M; ON)$ with equally spaced nodes, Optimization methods and operations research, applied mathematics, pp. 174–177, 191. Akad. Nauk SSSR Sibirsk. Otdel. Sibirsk. Energet. Inst., Irkutsk (1976). (in Russian)
- [16] L.F. Meyers, A. Sard, Best approximate integration formulas, J. Math. Physics, 29 (1950) 118–123.
- [17] S.M. Nikol’skii, To question about estimation of approximation by quadrature formulas, Uspekhi Matem. Nauk, 5:2 (36) (1950) 165–177. (in Russian)
- [18] S.M. Nikol’skii, Quadrature Formulas, Nauka, Moscow, 1988. (in Russian)
- [19] A. Sard, Best approximate integration formulas; best approximation formulas, Amer. J. Math. 71 (1949) 80–91.

- [20] A. Sard, Linear approximation, AMS, 1963.
- [21] I.J. Schoenberg, On monosplines of least deviation and best quadrature formulae, J. Soc. Indust. Appl. Math. Ser. B Numer. Anal. 2 (1965) 144–170.
- [22] I.J. Schoenberg, On monosplines of least square deviation and best quadrature formulae II. SIAM J. Numer. Anal. v.3, no. 2 (1966) 321–328.
- [23] I.J. Schoenberg, S.D. Silliman, On semicardinal quadrature formulae. Math. Comp. v.126 (1974) 483–497.
- [24] Kh.M. Shadimetov, Optimal formulas of approximate integration for differentiable functions, Candidate dissertation, Novosibirsk, 1983, p. 140. arXiv:1005.0163v1 [NA.math].
- [25] Kh.M. Shadimetov, Optimal quadrature formulas in $L_2^m(\Omega)$ and $L_2^m(R^1)$, Dokl. Akad. Nauk UzSSR. no. 3 (1983) 5–8. (in Russian).
- [26] Kh.M. Shadimetov. The discrete analogue of the differential operator d^{2m}/dx^{2m} and its construction, Questions of Computations and Applied Mathematics. Tashkent, (1985) 22-35. arXiv:1001.0556.v1 [math.NA] Jan. 2010.
- [27] Kh.M. Shadimetov, Optimal Lattice Quadrature and Cubature Formulas, Doklady Mathematics, v.63, no. 1 (2001) 92-94.
- [28] Kh.M. Shadimetov, Construction of weight optimal quadrature formulas in the space $L_2^{(m)}(0, N)$, Siberian J. Comput. Math. 5, no. 3 (2002) 275–293. (in Russian).
- [29] Kh.M. Shadimetov, A.R. Hayotov, Optimal quadrature formulas with positive coefficients in $L_2^{(m)}(0, 1)$ space, J. Comput. Appl. Math. 235 (2011) 1114–1128.
- [30] Kh.M. Shadimetov, A.R. Hayotov, Optimal quadrature formulas in the sense of Sard in $W_2^{(m, m-1)}$ space, Calcolo (2014) 51:211-243.
- [31] Kh.M. Shadimetov, A.R. Hayotov, S.S. Azamov, Optimal quadrature formula in $K_2(P_2)$ space, Applied Numerical Mathematics 62 (2012) 1893-1909.
- [32] Kh.M. Shadimetov, A.R. Hayotov, F.A. Nuraliev, On an optimal quadrature formula in Sobolev space $L_2^{(m)}(0, 1)$, Journal of Computational and Applied Mathematics 243 (2013) 91-112.
- [33] S.L. Sobolev, Introduction to the Theory of Cubature Formulas, Nauka, Moscow, 1974. (in Russian).
- [34] S.L. Sobolev, The coefficients of optimal quadrature formulas, Selected Works of S.L. Sobolev, Springer, (2006) 561–566.
- [35] S.L. Sobolev, V.L. Vaskevich, The Theory of Cubature Formulas, Kluwer Academic Publishers Group, Dordrecht, 1997.
- [36] F.Ya. Zagirova, On construction of optimal quadrature formulas with equal spaced nodes . Novosibirsk (1982), 28 p. (Preprint No. 25, Institute of Mathematics SD of AS of USSR). (in Russian).

- [37] Z.Zh. Zhamalov, Kh.M. Shadimetov, About optimal quadrature formulas (Russian), Dokl. Akademii Nauk UzSSR, 7 (1980) 3–5. (in Russian).
- [38] A.A. Zhensikbaev, Monosplines of minimal norm and the best quadrature formulas (Russian), Uspekhi Matem. Nauk, 36 (1981) 107–159. (in Russian).